

# Polylinear Additive Functionals of Superprocesses\*

E. B. Dynkin

*Department of Mathematics, Cornell University, White Hall, Ithaca, New York 14853*

Received October 12, 1996; accepted November 18, 1996

Let  $X$  be a superdiffusion in a domain  $E$  of  $\mathbb{R}^d$ . A polylinear additive functional of  $X$  corresponding to a positive Borel function  $\rho$  is given by the formula

$$A(B) = \int_B dt_1, \dots, dt_k \int_{E^k} \rho(t_1, z_1; \dots; t_k, z_k) X_{t_1}(dz_1) \cdots X_{t_k}(dz_k).$$

By a passage to the limit we extend this definition to a certain class of generalized

View metadata, citation and similar papers at [core.ac.uk](http://core.ac.uk)

$$\int v(dt_1, dz_1; \dots; dt_k, dz_k) v(dt'_1, dz'_1; \dots; dt'_k, dz'_k) \\ \times q^\mu(t_1, z_1; \dots; t_k, dz_k; t'_1, z'_1; \dots; t'_k, dz'_k) < \infty$$

for a certain kernel  $q^\mu$ . We prove that, if  $v\{t: t_i = s\} = 0$  for  $i = 1, \dots, k$  and for all  $s$ , then measure  $A_v$  has, a.s., the same property. This result is applicable, in particular, to self-intersection local times of  $X$ . © 1997 Academic Press

## 1. STATEMENT AND DISCUSSION OF THE RESULTS

**1.1. Polylinear Additive Functionals.** To every measurable space  $(E, \mathcal{B})$  there corresponds the space  $\mathcal{M}(E)$  of all finite measures on  $\mathcal{B}$ . If  $\xi = (\xi_t, \Pi_{r,x})$  is a Markov process in  $(E, \mathcal{B})$ , then, for a certain class  $\Psi$  of functions  $\psi$ , there exist branching processes in  $\mathcal{M}(E)$  with the branching mechanism described by  $\psi$ . More precisely, to every finite measure  $\mu$  on  $S = \mathbb{R}_+ \times E^1$  there corresponds a Markov process  $X = (X_t, P_\mu)$  [we call it a *superprocess over  $\xi$* ] such that, for all positive measurable functions  $f$ ,

$$P_\mu e^{-\langle f, X_t \rangle} = e^{-\langle f, u \rangle}, \quad (1.1)$$

where  $u$  is a unique solution of the equation

$$u(r, x) + \Pi_{r,x} \int_r^t \psi[u(s, \xi_s)] ds = \Pi_{r,x} f(t, \xi_t). \quad (1.2)$$

\* Partially supported by National Science Foundation Grant DMS-9623190.

<sup>1</sup> We put  $\mathbb{R}_+ = [0, \infty)$  and  $\mathbb{R}_{++} = (0, \infty)$ .

Our subject is additive functionals of order  $k$  of  $X$ . We assume that  $\psi(s, u) = u^2$  but the results can be extended to  $\psi \in \Psi$  which are  $2k$  times continuously differentiable in  $u$ .

All measures  $P_\mu$  are defined on a  $\sigma$ -algebra  $\mathcal{F}$  in  $\Omega$  such that  $X_t(B)$  is  $\mathcal{F}$ -measurable for every  $B \in \mathcal{B}$ . An *additive functional of order  $k$*  is a  $\Sigma$ -finite measure<sup>2</sup>  $A(\omega, \cdot)$  on the Borel  $\sigma$ -algebra in  $\mathbb{R}_{++}^k$  depending on parameter  $\omega \in \Omega$ . For every  $\mu \in \mathcal{M}(S)$  and every Borel set  $B \subset \mathbb{R}_{++}^k$ ,  $A(\cdot, B)$  is measurable relative to the completion of  $\mathcal{F}$  with respect to  $P_\mu$ . To every positive measurable function  $\rho$  on  $S^k$  there corresponds a functional

$$A_\rho(B) = \int_B dt \int_{E^k} \rho(t, z) X_{t_1}(dz_1) \cdots X_{t_k}(dz_k). \quad (1.3)$$

For the sake of brevity, we write  $t, z$  and  $dt$  for  $(t_1, \dots, t_k), (z_1, \dots, z_k)$  and  $dt_1 \cdots dt_k$ .

Put  $t \in T$  if all coordinates  $t_1, \dots, t_k$  are distinct. Functionals that we are interested in can explode outside  $T$ . A special role is played by a class  $\mathbb{B}$  of bounded subsets  $B$  of  $T$  characterized by the conditions: the closure  $\bar{B} \subset T$  and  $1_B(t^{(n)}) \rightarrow 1_B(t)$  as  $t^{(n)} \uparrow t$ .<sup>3</sup> For every  $\alpha, \beta > 0$ , the class  $\mathbb{B}$  contains the set  $T_{\alpha, \beta} = \{t: t_i \leq \alpha \text{ for } i = 1, \dots, k \text{ and } |t_i - t_j| \geq \beta \text{ for } i \neq j\}$  and the intersection of this set with every  $k$ -dimensional interval  $(a_1, b_1] \times \cdots \times (a_k, b_k]$ . A  $\sigma$ -finite measure on  $T$  is determined uniquely by its values on  $\mathbb{B}$ .

An additive functional of order  $k$  is called a *polylinear additive (PLA) functional* if it can be obtained by a passage to the limit from functionals of the form (1.3). More precisely, let  $\mathcal{M}^* \subset \mathcal{M}(S)$ . We say that  $A$  is a *special PLA functional with determining set  $\mathcal{M}^*$*  if there exists a sequence  $\rho_n$  such that  $A_{\rho_n}(B) \rightarrow A(B)$  in  $L^2(P_\mu)$  for all  $\mu \in \mathcal{M}^*$  and all  $B \in \mathbb{B}$ .  $A$  is a *PLA functional with determining set  $\mathcal{M}^*$*  if it is the sum of a countable family of special PLA functionals with determining set  $\mathcal{M}^*$ .

We assume that  $\xi$  is a right process in a Polish space  $E$  with transition function  $p(r, x; s, dy) = p(r, x; s, y) m(dy)$ , where  $m$  is a Radon measure. To simplify formulae we write  $dy$  for  $m(dy)$ . It is known (see [8]) that the density  $p(r, x; s, y)$  can be chosen to satisfy the condition<sup>4</sup>

$$\int p(r, x; s, y) dy p(s, y; t, z) = p(r, x; t, z) \quad (1.4)$$

<sup>2</sup> We say that a measure is  $\Sigma$ -finite if it is equal to the sum of a countable family of finite measures.

<sup>3</sup> This means coordinatewise monotone convergence:  $t_i^{(n)} \uparrow t_i$  for  $i = 1, \dots, k$ .

<sup>4</sup> No explicit indication of a domain under the integral sign means that the integral is taken over the entire domain of the corresponding measure.

for all  $r < s < t \in \mathbb{R}_+$  and all  $x, z \in E$ . In addition, we assume:

1.1.A. For every  $\beta > 0$  there exist a constant  $C(\beta)$  such that

$$p(r, x; s, y) \leq C(\beta) \quad \text{for all } 0 \leq r < s - \beta \quad \text{and for all } x, y \in E.$$

1.1.B. There exists a constant  $C$  such that

$$\int dx \, p(r, x; t, y) \leq C$$

for all  $r < t$  and all  $y$ .

We put  $p(r, x; t, y) = 0$  for  $r \geq t$ .

**1.2. Existence Theorem.** For every  $\mu \in \mathcal{M}(S)$  we introduce the functions

$$p^\mu(t, z) = \int \mu(dr, dx) \, p(r, x; t, z) \quad \text{on } S, \quad (1.5)$$

$$\begin{aligned} q^\mu(t, z; t' z') &= \int [1 + p^\mu(s, y)] \, ds \, dy \, p(s, y; t, z) \, p(s, y; t', z') \\ &\quad + [1 + p^\mu(t, z)][1 + p^\mu(t', z')] \quad \text{on } S \times S, \end{aligned} \quad (1.6)$$

and

$$q^\mu(t, z; t', z') = \sum_{j \in \mathcal{J}} \prod_{i=1}^k q^\mu(t_i, z_i; t_{j(i)}, z_{j(i)}) \quad \text{on } S^k \times S^k, \quad (1.7)$$

where  $\mathcal{J}$  is the set of all permutations of  $1, 2, \dots, k$ .

Let  $\mathbb{N}$  stand for the set of all  $\Sigma$ -finite measures  $\nu$  on  $T \times E^k$ . For every  $\nu, \nu' \in \mathbb{N}$ , we put

$$q^\mu(\nu, \nu') = \int q^\mu(t, z; t' z') \, \nu(dt, dz) \, \nu'(dt', dz'). \quad (1.8)$$

Denote by  $\nu_B$  the restriction of  $\nu$  to  $B \times E^k$  and by  $\nu^\Gamma$  its restriction to  $T \times \Gamma$  where  $\Gamma \subset E^k$ .

**THEOREM 1.1.** For every  $\nu \in \mathbb{N}$  and every  $\lambda = (\lambda_1, \dots, \lambda_k)$ , put

$$\rho_\nu^\lambda(t, z) = \int \left[ \prod_{i=1}^k \lambda_i e^{-\lambda_i(t'_i - t_i)} p(t_i, z_i; t'_i, z'_i) \right] \nu(dt', dz') \quad (1.9)$$

and denote by  $A_v^\lambda$  the functional corresponding to  $\rho_v^\lambda$  by formula (1.3). If the condition

$$q^\mu(v_B, v_B) < \infty \quad \text{for all } B \in \mathbb{B} \quad (1.10)$$

holds for every  $\mu \in \mathcal{M}^*$ , then there exists a [special] PLA functional  $A_v$  with determining set  $\mathcal{M}^*$  such that

$$A_v(B) = \lim_{\lambda \rightarrow \infty} A_v^\lambda(B) \quad \text{in } L^2(P_\mu) \quad (1.11)$$

for all  $\mu \in \mathcal{M}^*$  and all  $B \in \mathbb{B}$ . [Writing  $\lambda \rightarrow \infty$  means  $\lambda_i \rightarrow \infty$  for  $i = 1, \dots, k$ ,]  
If the condition

$$q^\mu(v_B^\Gamma, v_B^\Gamma) < \infty \quad \text{for all } B \in \mathbb{B} \quad \text{and all relatively compact } \Gamma \subset E^k \quad (1.12)$$

holds for all  $\mu \in \mathcal{M}^*$ , then there exists a PLA functional  $A_v$  with determining set  $\mathcal{M}^*$  such that

$$A_v = \sum A_{(v\Gamma_n)} \quad P_\mu\text{-a.s.} \quad (1.13)$$

for all  $\mu \in \mathcal{M}^*$  and for an arbitrary partition of  $E^k$  into disjoint relatively compact subsets  $\Gamma_n$ .

*Remark.* By the first part of Theorem 1.1,  $A_{(v\Gamma)}$  are special PLA functionals with determining set  $\mathcal{M}^*$ .

THEOREM 1.2. Put

$$\tilde{q}(t, z; t', z') = \int ds dy p(s, y; t, z) p(s, y; t', z') \quad \text{for } (t, z), (t', z') \in S \quad (1.14)$$

and

$$\tilde{q}(t, z; t', z') = \sum_{j \in \mathcal{J}} \prod_{i=1}^k \tilde{q}(t_i, z_i; t'_{j(i)}, z'_{j(i)}) \quad \text{for } (t, z), (t', z') \in S^k. \quad (1.15)$$

Formula (1.11) defines a special PLA functional with determining set  $\mathcal{M}^*$  if:

1.2.A. For all  $B \in \mathbb{B}$ ,

$$\tilde{q}(t, z; t', z') v_B(dt, dz) v_B(dt', dz') < \infty. \quad (1.16)$$

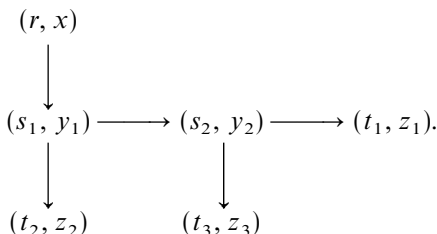
1.2.B. Function  $p^\mu(t, z)$  is bounded for all  $\mu \in \mathcal{M}^*$ .

Theorem 1.2 is an implication of Theorem 1.1 and a simple observation: if  $p^\mu(t, z) \leq C$  for all  $t, z$ , then

$$q^\mu(t, z; t', z') \leq [1 + C] \tilde{q}(t, z; t', z') + (1 + C)^2.$$

*Remark.* By 1.1.B,  $p^\mu(t, z)$  is bounded if  $\mu(dr, dx) = \rho(x) dx \gamma(dr)$  with bounded  $\rho$  and finite  $\gamma$ .

**1.3. Moments of PLA Functionals.** To describe moments of a PLA functional  $A_\nu$ , we use rooted labeled directed binary trees like the following one



The label of every vertex is an element of  $S$ . The label  $(r, x)$  is used for the root; leaves (or exits) are labeled by  $(t_i, z_i)$ . We denote by  $V_0$  the set of remaining vertices (which have multiplicities 3) and we label them  $(s_j, y_j)$ .

We write  $a: v \rightarrow v'$  if  $a$  is an arrow leading from vertex  $v$  to vertex  $v'$  and we put

$$p_a = p(u, w; u', w') \quad (1.17)$$

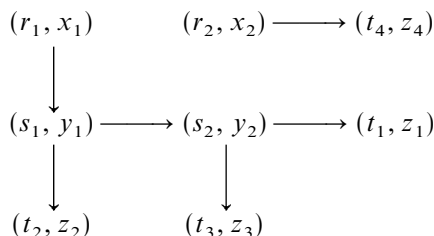
if  $(u, w)$  is the label of  $v$  and  $(u', w')$  is the label of  $v'$ . The product of  $p_a$  over all arrows  $a$  of a tree  $D$  is denoted by  $p_D$ . A special role is played by functions

$$L_D(r, x; t, z) = \int p_D \prod ds_j dy_j, \quad (r, x) \in S, \quad (t, z) \in S^k \quad (1.18)$$

obtained by integration of  $p_D$  over all the variables in the labels of  $v \in V_0$  and by

$$L_D^\mu(t, z) = \int \mu(dr, dx) L_D(r, x; t, z), \quad (t, z) \in S^k. \quad (1.19)$$

We call a *diagram* the union of a finite number of disjoint trees. [For instance the diagram



is the union of two trees.] Set

$$L_D = \prod_1^m L_{D_i}, \quad L_D^\mu = \prod_1^m L_{D_i}^\mu \quad (1.20)$$

for  $D = D_1 \cup \dots \cup D_m$ . The set of all diagrams with  $k$  exits is denoted by  $\mathbb{D}_k$ . [Diagrams obtained from each other by a permutation of labels of exits are considered as distinct elements of  $\mathbb{D}_k$ ; however, we do not distinguish diagrams obtained by relabeling of the roots or of the elements of  $V_0$ .] Put

$$L_D^\mu(v) = \int L_D^\mu(t, z) v(dt, dz). \quad (1.21)$$

We use notation  $\gamma_\varphi$  for the measure  $\varphi(w) \gamma(dw)$  where  $\gamma$  is a measure on a measurable space  $(W, \mathcal{W})$  and  $\varphi$  is a positive  $\mathcal{W}$ -measurable function.

**THEOREM 1.3.** *Let  $A_v$  be a PLA functional with determining set  $\mathcal{M}^*$  defined by (1.11) and (1.13) and let  $\mu \in \mathcal{M}^*$ . Then, for every positive Borel function  $\varphi$  on  $T$ ,*

$$P_\mu \int \varphi(t) A_v(dt) = \sum_{D \in \mathbb{D}_k} L_D^\mu(v_\varphi) \quad (1.22)$$

and for every positive Borel function  $\varphi$  on  $T^2$

$$P_\mu \int_{T^2} \varphi(t, t') A_v(dt) A_v(dt') = \sum_{D \in \mathbb{D}_{2k}} L_D^\mu[(v \times v)_\varphi]. \quad (1.23)$$

**1.4. Properties of  $A_v$ .** Denote by  $\mathbb{L}$  the family of all hyperplanes of the form  $L_a^i = \{t: t_i = a\}$ ,  $i = 1, \dots, k$ ,  $a > 0$ . For every measure  $\gamma$  on  $\mathbb{R}_+^k$ , we put  $L \in \mathbb{L}(\gamma)$  if  $L \in \mathbb{L}$  and  $\gamma(L) > 0$ . We say that  $\gamma$  is *diffuse* if  $\mathbb{L}(\gamma) = \emptyset$ .<sup>5</sup>

<sup>5</sup> For a measure  $\gamma$  on  $\mathbb{R}_+$  this definition (which means  $\gamma(a) = 0$  for all  $a$ ) is the conventional definition of a diffuse measure. In general,  $\gamma$  is diffuse if and only if its projections on all coordinate axes are diffuse measures.

If  $\nu$  is a measure on  $\mathbb{R}_+^k \times E^k$ , then  $\mathbb{L}(\nu)$  means  $\mathbb{L}(\gamma)$  where  $\gamma$  is the projection of  $\eta$  on  $\mathbb{R}_+^k$  [i.e.,  $\gamma(B) = \nu(B \times E^k)$ ]. A measure  $\nu$  is called diffuse if  $\gamma$  is diffuse. All measures  $\nu(dt, dz) = dt \eta(dz)$  are diffuse.

**THEOREM 1.4.** *Let  $A_\nu$  be a PLA functional with determining set  $\mathcal{M}^*$  defined by (1.11) and (1.13) and let  $\mu \in \mathcal{M}^*$ . We have: (a) If  $\nu(B \times E^k) = 0$ , then  $A_\nu(B) = 0$   $P_\mu$ -a.s. (b) If  $\nu$  is diffuse, then  $A_\nu$  is diffuse  $P_\mu$ -a.s. (c)  $\mathbb{L}(A_\nu) \subset \mathbb{L}(\nu)$   $P_\mu$ -a.s.*

**1.5. PLA Functionals of Superdiffusions.** A superdiffusion  $X$  is a superprocess corresponding to a diffusion  $\xi$  in a domain  $E$  of  $\mathbb{R}^d$ . The transition function  $p(r, x; t, y)$  of  $\xi$  satisfies conditions 1.1.A-B and the following condition:

1.5.A. For every  $\alpha > 0$ , there exist constants  $K, \lambda$  such that, for all  $0 \leq r < t \leq \alpha$ ,

$$p(r, x; t, y) \leq K \pi_{t-s}[\lambda(y-x)] = K \pi_{(t-s)/\lambda}(y-x), \quad (1.24)$$

where

$$\pi_t(x) = (2\pi t)^{-d/2} e^{-|x|^2/(2t)} \quad (1.25)$$

[see, e.g., [10], Chapter 1, Theorem 11].

Put

$$\hat{q}_u(x) = \int_0^u t \pi_t(x) dt. \quad (1.26)$$

**THEOREM 1.5.** *Suppose that*

$$\nu(dt, dz) = \eta(dz) dt, \quad (1.27)$$

where  $\eta$  is a finite measure on  $E^k$  such that

$$\int \eta(dz) \left[ \prod_1^k \hat{q}_u(z_i - z'_{j(i)}) \right] \eta(dz') < \infty \quad (1.28)$$

for all  $u > 0$ ,  $j \in \mathcal{J}$ . If  $\mathcal{M}^*$  satisfies Condition 1.2.B, then Formula (1.11) defines a special PLA functional of superdiffusion  $X$  with determining set  $\mathcal{M}^*$ .

Indeed,  $\pi_t(x) = \pi_t(-x)$  and

$$\int \pi_s(y) dy \pi_t(z-y) = \pi_{s+t}(z)$$

for all  $s, t > 0, z \in E$ . Therefore, by (1.14) and (1.24),

$$\begin{aligned} & \int_0^\alpha dt \int_0^\alpha dt' \tilde{q}(t, z; t', z') \\ & \leq (K\lambda)^2 \int_0^\alpha ds \int_0^\alpha dt_1 \int_0^\alpha dt_2 \pi_{t_1+t_2}(z-z') \leq (K\lambda)^2 \alpha \hat{q}_{2\alpha/\lambda}(z-z') \end{aligned} \quad (1.29)$$

and Theorem 1.5 follows from Theorem 1.2.

**1.6. Local Times.** The *local time*  $L_c$  for  $X$  at point  $c \in E$  is the PLA functional of order 1 corresponding, by Theorem 1.5 to

$$\nu(dt, dz) = \delta_c(dz) dt, \quad (1.30)$$

where  $\delta_c$  is the unit mass concentrated at point  $c$ . For every Borel function  $f$  on  $E^2$ ,

$$\int f(z, z') \delta_c(dz) \delta_c(dz') = f(c, c).$$

Suppose that  $X$  is a superdiffusion. Then, by (1.26) and (1.25),

$$\int \hat{q}_u(z-z') \delta_c(dz) \delta_c(dz') = \hat{q}_u(0) = \int_0^u t(2\pi t)^{-d/2} dt.$$

If  $d \leq 3$ , then condition (1.28) holds for  $\eta = \delta_c$  and therefore  $L_c$  is a PLA functional with determining set  $\mathcal{M}^*$  for every set  $\mathcal{M}^*$  subject to Condition 1.2.B.

The *self-intersection local time*  $L^k$  of order  $k > 1$  is the PLA functional of order  $k$  corresponding, by Theorem 1.5 to

$$\nu(dt, dz) = \delta^k(dz) dt, \quad (1.31)$$

where  $\delta^k$  is the image of the reference measure  $m$  under the mapping  $i(z) = (z, \dots, z)$  from  $E$  to  $E^k$ . For every positive Borel function  $f$  on  $E^k \times E^k$ ,

$$\int f(z, z') \delta^k(dz) \delta^k(dz') = \int_E dz \int_E dz' f(z, \dots, z; z', \dots, z').$$

Suppose that  $X$  is a superdiffusion. Condition (1.28) holds for  $[\delta^k]^r$  if

$$\int_{\Gamma^*} dz \int_{\Gamma^*} dz' \hat{q}_u(z-z')^k < \infty \quad \text{for all } u,$$

where  $\Gamma^* = i^{-1}(\Gamma)$ .



For a relatively compact set  $\Gamma$ , this is equivalent to the condition

$$\int_{\Gamma^*} \hat{q}_u(x)^k dx < \infty \quad \text{for all } u. \quad (1.32)$$

For  $d \geq 3$ ,  $\hat{q}_u(x) \leq \text{const.} \int_0^u t^{-1/2} dt < \infty$  and therefore (1.32) holds for all  $k$ . Change of variables  $s = x^2/2t$  in (1.26) yields

$$\hat{q}_u(x) = \text{const.} |x|^{4-d} \Phi_d(x^2/2u), \quad (1.33)$$

where

$$\Phi_d(t) = \int_t^\infty s^{d/2-3} e^{-s} ds.$$

For  $d \geq 5$ ,  $\Phi_d(t) \leq \Phi_d(0) < \infty$  and  $\hat{q}_u(x) \leq \text{const.} |x|^{4-d}$ . Condition (1.32) holds for  $k \leq 4$  if  $d=5$  and for  $k \leq 2$  if  $d=6$  or  $7$ . It is satisfied for all  $k$  if  $d=4$  because  $\Phi_4(t) \leq \text{const.} [|\log t| + 1]$ .

**1.7. Time-Homogeneous Processes.** If the transition function of  $\xi$  is stationary, then the density function  $p(r, x; s, y)$  subject to Condition (1.4) can be chosen to depend only on the difference  $s - r$  (see [1], Lemma 2.1). We put  $p_t(x, y) = p(r, x; r + t, y)$ . A superprocess corresponding to  $\xi$  can be constructed in such a way that

$$\theta_u X_t = X_{t+u}; \quad P_{\kappa_u(\mu)}(\theta_u Y) = P_\mu Y, \quad (1.34)$$

where  $\theta_u$  are the so-called shift operators and  $\kappa_u$  are transformations of  $S$  given by the formula  $\kappa_u(t, z) = (t + u, z)$ . To every measure  $\mu \in \mathcal{M}(E)$  there corresponds a measure  $\bar{\mu}$  on  $S$  which is the image of  $\mu$  under the mapping  $x \rightarrow (0, x)$  from  $E$  to  $S$ . We put  $P_\mu = P_{\bar{\mu}}$ . Let

$$g_u(x, y) = \int_0^u p_t(x, y) dt, \quad (1.35)$$

$$g_u^\mu(y) = \int_E \mu(dx) g_u(x, y) \quad \text{for } y \in E, \quad (1.36)$$

$$g_u^\mu(z) = \prod_{i=1}^k g_u^\mu(z_i) \quad \text{for } z \in E^k, \quad (1.37)$$

$$G_u^\mu(z, z') = \int [u + g_u^\mu(y)] dy g_u(y, z) g_u(y, z') + [u + g_u^\mu(z)][u + g_u^\mu(z')]. \quad (1.38)$$

THEOREM 1.6. For every measure  $\eta$  on  $E^k$  and every  $\lambda = (\lambda_1, \dots, \lambda_k)$ , put

$$\rho_\eta^\lambda(z) = \int \left[ \prod_1^k \lambda_i g^{\lambda_i}(z, z') \right] \eta(dz'), \quad (1.39)$$

where

$$g^\lambda(x, y) = \int_0^\infty e^{-\lambda t} p_t(x, y) dt \quad \text{for } x, y \in E. \quad (1.40)$$

Let

$$A_\eta^\lambda(B) = \int_B dt \int_{E^k} \rho_\eta^\lambda(z) X_{t_1}(dz_1) \cdots X_{t_k}(dz_k). \quad (1.41)$$

Suppose  $\mathcal{M}^* \subset \mathcal{M}(E)$  is such that, for every  $\mu \in \mathcal{M}^*$ ,

$$\int \eta(dz) G_u^\mu(z, z') \eta(dz') < \infty \quad (1.42)$$

for all  $u > 0$ . Then there exists a PLA functional  $A_\eta$  with determining set  $\mathcal{M}^*$  subject to the conditions:

1.7.A. For each  $\mu \in \mathcal{M}^*$ ,

$$A_\eta(B) = \lim A_\eta^\lambda(B) \quad \text{in } L^2(P_\mu) \quad (1.43)$$

for all  $B \in \mathbb{B}$ .

1.7.B. For every Borel set  $B \subset \mathbb{R}_{++}^k$  and every  $u > 0$ ,

$$\theta_u A_\eta(B) = A_\eta(\kappa_u(B)) \quad P_{\kappa_u(\mu)}\text{-a.s.} \quad (1.44)$$

for all  $\mu \in \mathcal{M}^*$ .

*Proof.* Note that function defined by (1.39) coincides with function given by (1.9) with  $v(dt, dz) = dt \eta(dz)$  and therefore  $A_\eta^\lambda$  defined by (1.41) is identical to  $A_v^\lambda$  in Theorem 1.1.

It follows from (1.5)–(1.8) and (1.38) that

$$q^\mu(v_B, v_B) \leq \int \eta(dz) G_u^\mu(z, z') \eta(dz') \quad (1.45)$$

if  $B \subset T_{u, \beta}$ . The existence of a PLA subject to Condition 1.7.A follows from Theorem 1.1. It remains to prove that it satisfies 1.7.B.

By (1.34),

$$\theta_u A_\eta^\lambda(B) = A_\eta^\lambda(\kappa_u(B)) \quad P_\mu\text{-a.s.} \quad (1.46)$$

for all  $\mu$  and  $B$ .

Since  $\kappa_u(v) = v$ , we have

$$q^{\kappa_u(\mu)}(v_B, v_B) = q^\mu(v_{\kappa_u(B)}, v_{\kappa_u(B)})$$

and, since class  $\mathbb{B}$  is invariant relative to  $\kappa_u$ , Condition (1.10) holds for  $\kappa_u(\mu)$  if it holds for  $\mu$  and, by (1.11),

$$P_{\kappa_u(\mu)}[A_\eta^\lambda(\kappa_u(B)) - A_\eta(\kappa_u(B))]^2 \rightarrow 0 \quad \text{as } \lambda \rightarrow \infty. \quad (1.47)$$

On the other hand, by (1.34) and (1.11)

$$P_{\kappa_u(\mu)}\{\theta_u[A_\eta^\lambda(B) - A_\eta(B)]^2 = P_\mu[A^\lambda(B) - A_\eta(B)]^2 \rightarrow 0 \quad \text{as } \lambda \rightarrow \infty. \quad (1.48)$$

Property 1.7.B follows from (1.46), (1.47) and (1.48). ■

Moments of  $A_\eta$  can be expressed in terms of the Green's function

$$g(x, y) = \int_0^\infty p_t(x, y) dt \quad (1.49)$$

and diagrams labeled by points of  $E$  (not of  $S$ ). We set  $g_a(x, y)$  for an arrow  $a: x \rightarrow y$  and we denote by  $g_D$  the product of  $g_a$  over all arrows  $a$  of  $D$ . Put

$$\tilde{L}_D(x, z) = \int g_D \prod dy_j, \quad \tilde{L}^\mu(z) = \int \mu(dx) \tilde{L}_D(x, z) \quad \text{for } \mu \in \mathcal{M}(E) \quad (1.50)$$

for a labeled tree  $D$  (cf. (1.18)) and

$$\tilde{L}_D = \prod_1^m \tilde{L}_{D_i}, \quad \tilde{L}_D^\mu = \prod_1^m L_{D_i} \quad (1.51)$$

for the union  $D$  of trees  $D_1, \dots, D_m$ . If  $v$  is given by (1.27) and if  $\mu \in \mathcal{M}(E)$ , then

$$L_D^\mu(v) = \int \tilde{L}^\mu(z) \eta(dz). \quad (1.52)$$

## 2. PROOFS OF THEOREM 1.1 AND 1.3

2.1. Theorems 1.1 and 1.3 are immediate of the following lemmas.

LEMMA 2.1. *Let  $A_n$  be additive functionals of order  $k$  of  $X$  with determining set  $\mathcal{M}^*$  and let  $\lim A_n(B)$  in  $L^2(P_\mu)$  exists for all  $\mu \in \mathcal{M}^*$  and all  $B \in \mathbb{B}$ . Then there exists an additive functional  $A$  of  $X$  with determining set  $\mathcal{M}^*$  such that  $P_\mu[A_n(B) - A(B)]^2 \rightarrow 0$  for all  $\mu \in \mathcal{M}^*$ ,  $B \in \mathbb{B}$ .*

LEMMA 2.2. *If*

$$\int \prod_1^k [P^\mu(t_i, z_i)] v_B(dt, dz) < \infty, \quad (2.1)$$

*then  $L_D^\mu(v_B) < \infty$  for all  $D \in \mathbb{D}_k$ . Condition (1.10) implies that  $L_D^\mu(v_B \times v_B) < \infty$  for all  $B \in \mathbb{B}$ ,  $D \in \mathbb{D}_{2k}$ .*

LEMMA 2.3. *Let  $A_v^\lambda$  be the PLA functional of  $X$  described in Theorem 1.1. Condition (2.1) implies*

$$\lim_{\lambda \rightarrow \infty} P_\mu A_v^\lambda(B) = \sum_{D \in \mathbb{D}_k} L_D^\mu(v_B) < \infty \quad \text{for all } B \in \mathbb{B}. \quad (2.2)$$

*Condition (1.10) implies*

$$\lim_{\lambda, \lambda' \rightarrow \infty} P_\mu A_v^\lambda(B) A_v^{\lambda'}(B') = \sum_{D \in \mathbb{D}_{2k}} L_D^\mu(v_B \times v_{B'}) < \infty \quad \text{for all } B, B' \in \mathbb{B}. \quad (2.3)$$

To prove Theorem 1.1, we note that, by (2.3),

$$\lim_{\lambda, \lambda' \rightarrow \infty} P_\mu [A^\lambda(B) - A^{\lambda'}(B)]^2 = 0$$

and the existence of  $A_v$  subject to (1.11) follows from Lemma 2.1.

To get Theorem 1.3, we note that Formula (1.22) for  $\varphi = 1_B$  and Formula (1.23) for  $\varphi = 1_{B \times B'}$  follow from (1.11) and (2.2). The extension to general  $\varphi$  is routine.

The rest of Section 2 is devoted to proving Lemmas 2.1–2.3.

2.2. *Proof of Lemma 2.1.* 1°. For every finite set  $A = \{0 = s_0 < s_1 < \dots < s_n\}$ , we put  $\Delta_i = (s_{i-1}, s_i]$ ,  $i = 1, 2, \dots, n$  and we denote by  $B_A$  the union of  $\Delta_{i_1} \times \Delta_{i_2} \times \dots \times \Delta_{i_k}$  over all  $i_1, i_2, \dots, i_k$  such that  $|i_a - i_b| > 1$  for all  $a \neq b$ . Note that  $B_A \subset B_{\lambda'}$  if  $A \subset A'$ .

Let  $A_1 \subset \dots \subset A_m \subset \dots$  and let the union of  $A_m$  be everywhere dense in  $\mathbb{R}_+$ . Then every  $B \in \mathbb{B}$  is contained in  $B_{A_n}$  for sufficiently large  $n$ .

2°. Formula  $A_n^m(B) = A_n(B \cap B_{A_m})$  defines an additive functional which is finite  $P_\mu$ -a.s. for every  $\mu \in \mathcal{M}^*$ . For every  $\mu \in \mathcal{M}^*$  and every  $k$ -dimensional interval  $B = (a_1, b_1] \times \cdots \times (a_k, b_k]$  there exists  $\lim A_n^m$  in  $L^2(P_\mu)$  as  $n \rightarrow \infty$ . Therefore (see, e.g., the Appendix in [7])<sup>6</sup> there exists an additive functional  $A^m$  of order  $k$  such that, for all  $\mu \in \mathcal{M}^*$  and all  $k$ -dimensional intervals  $B$ ,  $A_n^m(B) \rightarrow A^m(B)$  in  $L^2(P_\mu)$  as  $n \rightarrow \infty$  (cf. proof of Theorem 5.1 in [2]).

Since  $A_n^m(B) = A_n^{m+1}(B)$  for  $B \subset B_{A_m}$ , we conclude that measures  $A^m$  and  $A^{m+1}$  coincide,  $P_\mu$ -a.s., on  $B_{A_m}$ . There exists a measure  $A$  which is equal,  $P_\mu$ -a.s., to  $A^m$  on  $B_{A_m}$  for all  $m$ . Clearly, it satisfies conditions stated in our lemma. ■

**2.3.** Proofs of both parts of Lemma 2.2 are similar. We prove the second part and leave the proof of the first part (which is much easier) to the reader. We start with necessary preparations. By (1.21),  $L_D^\mu(\nu_B \times \nu_B)$  is equal to the integral of

$$L_D^\mu(t_1, z_1; \dots; t_k, z_k; t'_1, z'_1; \dots; t'_k, z'_k) \quad (2.4)$$

with respect to  $\nu_B \times \nu_B$ . For some  $\alpha, \beta > 0$ , measure  $\nu_B$  is concentrated on  $T_{\alpha, \beta}$  and therefore we can restrict ourselves by the values

$$\begin{aligned} t_i, t'_i &\leq \alpha & \text{for } i = 1, \dots, k; \\ |t_i - t_j| &\geq \beta, & |t'_i - t'_j| \geq \beta \quad \text{for } i \neq j. \end{aligned} \quad (2.5)$$

The arguments in (2.4) are the labels of roots and leaves in  $D \in \mathbb{D}_{2k}$ . We paint the leaves  $(t_i, z_i)$  in green and the leaves  $(t'_i, z'_i)$  in yellow. We also paint each arrow in white or red. We say that a vertex  $v$  is red if it belongs to three red arrows. Let  $\mathbb{D}_{2k}^*$  stand for the set of colored diagrams. For every  $D \in \mathbb{D}_{2k}^*$  we define  $L_D$  by Formula (1.18) modified as follows:

- (a) if an arrow  $a$  is red, then  $p_a = 1_{u < u'}$  (for white  $a$ ,  $p_a$  is defined by (1.17) as before);
- (b) integration with respect to  $dy_i$  is dropped if  $(s_i, y_i)$  is a red vertex.

We say that  $D \in \mathbb{D}_{2k}^*$  is dominated by a family  $\mathbb{D}' \subset \mathbb{D}_{2k}^*$  if there exists a constant  $K$  such that

$$L_D \leq K \sum_{D' \subset \mathbb{D}'} L_{D'}$$

for all values of arguments subject to Conditions (2.5).

<sup>6</sup> In [7] the case  $k = 1$  is considered but only minor modifications are needed to cover the case  $k > 1$ .

**PROPOSITION 2.1.** *If a vertex  $v$  belongs to two red arrows and one white arrow  $a$ , then  $D$  is dominated by  $D'$  obtained from  $D$  by painting arrow  $a$  red.*

*Proof.* The expressions for  $L_D$  and  $L_{D'}$  differ only by the values of  $p_a$ . Suppose that  $a = (s, y) \rightarrow (s', y')$ . Since  $\int p(s, y; s', y') dy' \leq 1_{s < s'}$  and, by 1.1.B,  $\int p(s, y; s', y') dy \leq C 1_{s < s'}$ , we have  $L_D \leq (C \vee 1) L_{D'}$ . ■

We say that  $\pi: v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_l$  is a white path from  $v$  to  $e$  if  $v = v_1$ ,  $e = v_l$  and if all arrows  $a_i = v_i \rightarrow v_{i+1}$  are white.

**PROPOSITION 2.2.** *Let  $\pi: v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_l$  and  $\pi': v'_1 \rightarrow v'_2 \rightarrow \dots \rightarrow v'_m$  be two white paths from  $v$  to two distinct exits  $e$  and  $e'$ . Denote by  $D_i$  the diagram obtained from  $D$  by changing color of arrow  $a_i = v_i \rightarrow v_{i+1}$  and by  $D'_i$  the diagram obtained from  $D$  by an analogous operation on  $a'_i = v'_i \rightarrow v'_{i+1}$ . If  $e$  and  $e'$  have the same color, then  $D$  is dominated by the family  $\mathbb{D}' = \{D_1, \dots, D_{l-1}, D'_1, \dots, D'_{m-1}\}$ .*

*Proof.* Let  $s_i$  and  $s'_j$  be the time variables in the labels of  $v_i$  and  $v'_j$ . Note that  $s_1 = s'_1$  and therefore

$$(s_2 - s_1) + \dots + (s_l - s_{l-1}) - (s'_2 - s'_1) - \dots - (s'_m - s'_{m-1}) = s_l - s'_m. \quad (2.6)$$

By (2.5),  $|s_l - s'_m| \geq \beta$  and therefore at least one difference in (2.6) is bigger than or equal to  $\beta/(l+m-2)$ . The Proposition 2.2 follows from 1.1.A. ■

**2.4. Proof of Lemma 2.2.** It is sufficient to prove that  $L_D^\mu(v_B \times v_B) < \infty$  for  $D \in \mathbb{D}_{2k}^*$  not dominated by any family  $D' \subset \mathbb{D}_{2k}^*$ . By Proposition 2.1, for such a diagram, the end of every white arrow is the beginning of another white arrow (unless it is an exit). The beginning of every white arrow belongs to another white arrow (unless it is a root).

If a vertex  $v$  is not the end of a red arrow, then there exists a unique maximal white path  $\pi: v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_m$  such that  $v_m = v$ . By Proposition 2.2, the maximal white paths leading to exits of the same color are disjoint. Let  $\pi_i$  be the maximal path leading to exit  $(t_i, z_i)$  and  $\pi'_j$  be the maximal path leading to  $(t'_j, z'_j)$ . For every  $\pi_i$  there exists at most one  $\pi'_j$  which is not disjoint from  $\pi_i$ . Put  $i \in I, j(i) = j$  if  $\pi_i$  and  $\pi'_j$  intersect. Let  $i \in J$  if  $\pi_i$  and  $\pi'_j$  are disjoint for all  $j$  and let  $i \in J'$  if  $\pi'_i$  is disjoint from all  $\pi_i$ . [The exits  $(t_i, z_i)$  with  $i \notin I \cup J$  and  $(t'_j, z'_j)$  with  $j \notin j(I) \cup J'$  are the ends of red arrows.]

For  $i \in I$ , the common part of  $\pi_i$  and  $\pi'_{j(i)}$  is either a path  $\rho$  starting from a root or it is a single vertex which is not a root. We put  $i \in I_1$  in the first case and  $i \in I_2$  in the second case. Every white arrow belongs to at least one of paths  $\pi_i, \pi'_i, i = 1, \dots, k$ . Therefore the white arrows of the diagram  $D$  form a diagram  $D'$  which looks like

$$\begin{array}{ccccc}
 & & (r_i, x_i) & & \\
 & & \downarrow \rho_i & & \\
 (t_i, z_i) & \xleftarrow{\sigma_i} & (s, y) & \xrightarrow{\sigma'_{j(i)}} & (t'_{j(i)}, z'_{j(i)})
 \end{array}$$

for  $i \in I_1$ ,

$$(t_i, z_i) \xleftarrow{\pi_i} (s, y) \xrightarrow{\pi'_{j(i)}} (t'_{j(i)}, z'_{j(i)})$$

for  $i \in I_2$ ,

$$(r_i, x_i) \xrightarrow{\pi_i} (t_i, z_i)$$

for  $i \in J$ ,

$$(r_i, x_i) \xrightarrow{\pi'_i} (t'_i, z'_i)$$

for  $i \in J'$ . (In the first row, the path  $\pi_i$  consists of  $\rho_i$  and  $\sigma_i$  and  $\pi'_i$  consists of  $\rho_i$  and  $\sigma'_{j(i)}$ .)

We associate with a path  $\pi: (s_1, y_1) \rightarrow (s_2, y_2) \rightarrow \dots \rightarrow (s_l, y_l)$  a function

$$Q_\pi(s_1, y_1; s_l, y_l) = \int \prod_{i=2}^l p(s_{i-1}, y_{i-1}; s_i, y_i) \prod_{i=2}^{l-1} ds_i dy_i.$$

[Here  $l \geq 2$  and  $Q_\pi = p(s_1, y_1; s_2, y_2)$  if  $l=2$ .] Since all  $s_i$  belong to interval  $[0, \alpha]$ , (1.4) implies

$$Q_\pi(s_1, y_1; s_l, y_l) \leq \alpha^{l-2} p(s_1, y_1; s_l, y_l). \quad (2.7)$$

If  $(s_1, y_1)$  is a root, then we put

$$Q_\pi^\mu(s_l, y_l) = \int \mu(ds_1, dy_1) Q_\pi(s_1, y_1; s_l, y_l). \quad (2.8)$$

We associate with diagram  $D'$  an expression

$$\Phi_{D'} = \prod_{i \in I \cup J} \Phi_i \prod_{j \in J'} \Phi'_j, \quad (2.9)$$

where

$$\Phi_i = \begin{cases} \int Q_{\rho_i}^\mu(s, y) ds dy Q_{\sigma_i}(s, y; t_i, z_i) Q_{\sigma'_{j(i)}}(s, y; t'_{j(i)}, z'_{j(i)}) & \text{for } i \in I_1, \\ \int ds dy Q_{\pi_i}(s, y; t_i, z_i) Q_{\pi'_{j(i)}}(s, y; t'_{j(i)}, z'_{j(i)}) & \text{for } i \in I_2, \\ Q_{\pi_i}^\mu(t_i, z_i) & \text{for } i \in J, \end{cases} \quad (2.10)$$

$$\Phi'_i = Q_{\pi'_i}^\mu(t'_i, z'_i) \quad \text{for } i \in J'. \quad (2.11)$$

We continue the map  $j: I \rightarrow \{1, \dots, k\}$  to a permutation of  $\{1, \dots, k\}$  such that  $j(i) = i$  for  $i \in J \cup J'$ . Note that

$$\Phi_{D'} = \prod_{i=1}^k \tilde{\Phi}_i,$$

where

$$\tilde{\Phi}_i = \begin{cases} \Phi_i & \text{for } i \in I, \\ \Phi_i \Phi'_i = \Phi_i \Phi'_{j(i)} & \text{for } i \in J \cap J', \\ \Phi_i = \Phi_{j(i)} & \text{for } i \in J \setminus J', \\ \Phi'_i = \Phi'_{j(i)} & \text{for } i \in J' \setminus J, \\ 1 & \text{for } i \notin I \cup J \cup J'. \end{cases} \quad (2.12)$$

By (2.7) and (1.6),  $\tilde{\Phi}_i \leq c' q^\mu(t_i, z_i; t'_{j(i)}, z'_{j(i)})$  for all  $i$  (constant  $c'$  depends only on  $\alpha$ ). By comparing the definition (1.18)–(1.19) of  $L_D^\mu$  and the definition (2.9)–(2.10) of  $\Phi_{D'}$  we note that

$$L_D^\mu(t, z; t', z') \leq c'' \Phi_{D'}, \quad (2.13)$$

where constant  $c''$  depends only on  $\alpha$  and  $\mu$ . By (1.21) and (1.8), this implies  $L_D^\mu(v_B \times v_B) \leq c'' q^\mu(v_B, v_B)$  which proves the second statement of Lemma 2.2. ■

**2.5. Proof of Lemma 2.3.** It follows from Theorem 1.1' in [4] that

$$P_\mu \int \rho(t, z) X_{t_1}(dz_1) \cdots X_{t_k}(dz_k) = \sum_{D \in \mathbb{D}_k} \int_{E^k} L_D^\mu(t, z) \rho(t, z) dz. \quad (2.14)$$

Put

$$v^\lambda(dt, dz) = \rho_v^\lambda(t, z) dt dz, \quad (2.15)$$

where  $\rho_v^\lambda$  is defined by (1.9). Formula (2.14) implies

$$\begin{aligned} P_\mu A_v^\lambda(B) &= \sum_{D \in \mathbb{D}_k} L_D^\mu(v_B^\lambda), \\ P_\mu A_v^\lambda(B) A_v^{\lambda'}(B') &= \sum_{D \in \mathbb{D}_{2k}} L_D^\mu(v_B^\lambda \times v_{B'}^{\lambda'}), \end{aligned} \quad (2.16)$$

where  $v_B^\lambda$  is the restriction of  $v^\lambda$  to  $B \times E^k$ . Lemma 2.3 will be proved if we show that, under Condition (2.1),

$$L_D^\mu(v_B^\lambda) \rightarrow L_D^\mu(v_B) \quad \text{as } \lambda \rightarrow \infty \quad (2.17)$$



and, under Condition (1.10),

$$L_D^\mu(v_B^\lambda \times v_{B'}^{\lambda'}) \uparrow L_D^\mu(v_B \times v_{B'}) \quad \text{as } \lambda, \lambda' \rightarrow \infty. \quad (2.18)$$

[The limits in (2.17) and (2.18) are finite by Lemma 2.2.] We prove (2.17). Only minor modifications are needed to prove (2.18).

It follows from (1.17)–(1.21) that there exists a finite measure  $M_D^\mu$  on  $S^k$  such that, for all  $v$ ,

$$L_D^\mu(v) = \int M_D^\mu(ds, dy) \left[ \prod_{i=1}^k p(s_i, y_i; t_i, z_i) \right] v(dt, dz) \quad (2.19)$$

(each factor  $p(s_i, y_i; t_i, z_i)$  corresponds to an arrow leading to an exit in  $D$ ). In particular,

$$L_D^\mu(v_B^\lambda) = \int M_B^\mu(ds, dy) F^\lambda(s, y), \quad (2.20)$$

where

$$F^\lambda(s, y) = \int \rho_v^\lambda(t, z) 1_B(t) \left[ \prod_1^k p(s_i, y_i; t_i, z_i) \right] dt dz. \quad (2.21)$$

By (1.4) and (1.9),

$$F^\lambda(s, y) = \varphi^\lambda(s, t') \left[ \prod_1^k p(s_i, y_i; t'_i, z'_i) \right] v(dt', dz'), \quad (2.22)$$

where

$$\begin{aligned} \varphi^\lambda(s, t') &= \int \left[ \prod_1^k 1_{s_i < t_i < t'_i} \lambda_i e^{-\lambda_i(t'_i - t_i)} \right] 1_B(t) dt \\ &= \int \left[ \prod_1^k 1_{0 < r_i < \lambda_i(t'_i - s_i)} dr_i e^{-r_i} \right] 1_B[\tau^\lambda(t', r)] \end{aligned} \quad (2.23)$$

with

$$\tau^\lambda(t', r) = (t'_1 - r_1/\lambda_1, \dots, t'_k - r_k/\lambda_k).$$

By the definition of class  $\mathbb{B}$ ,  $1_B[\tau^\lambda(t', r)] \rightarrow 1_B(t')$  as  $\lambda \rightarrow \infty$ . Therefore  $\varphi^\lambda(s, t') \rightarrow 1_B(t') \prod_1^k 1_{s_i < t'_i}$ . Since  $\varphi^\lambda(s, t') \leq 1$ , Formula (2.17) follows from (2.20), (2.22) and the dominated convergence theorem. ■

## 3. PROOF OF THEOREM 1.4

**3.1. Preliminaries.** Measures  $\gamma$  and  $\nu$  considered in this section assumed to be  $\Sigma$ -finite and therefore the corresponding families  $\mathbb{L}(\gamma)$  and  $\mathbb{L}(\nu)$  are at most countable. Denote by  $\mathcal{L}(\gamma)$  the union of all sets  $L \in \mathbb{L}(\gamma)$ .

**LEMMA 3.1.** *A set  $L \in \mathbb{L}$  belongs to  $\mathbb{L}(\gamma)$  if and only if  $L \subset \mathcal{L}(\gamma)$ .*

*Proof.* For every  $i$ , the set

$$A_i = \{s : \gamma(L_s^i) > 0\} \quad (3.1)$$

is at most countable. Let us prove that, if  $L_a^i \subset \mathcal{L}(\gamma)$ , then  $L \in \mathbb{L}(\gamma)$ . Indeed, there exists  $t \in L_a^i$  such that  $t_j \notin A_j$  for all  $j \neq i$ . Since  $t \in \mathcal{L}(\gamma)$  and since it does not belong to any  $L_s^j \in \mathbb{L}(\gamma)$  with  $j \neq i$ , it must be an element of a set  $L_s^i \in \mathbb{L}(\gamma)$ . Since  $L_a^i$  and  $L_s^i$  are not disjoint, they coincide and  $L_a^i = L_s^i \in \mathbb{L}(\gamma)$ . The rest of the lemma is trivial. ■

**3.2. Put**

$$J_\gamma = \int 1_C(t, t') \gamma(dt) \gamma(dt'), \quad (3.2)$$

where

$$C = \{(t, t') : t_i = t'_i \text{ for some } i = 1, \dots, k\}. \quad (3.3)$$

**LEMMA 3.2.** *Measure  $\gamma$  is diffuse if and only if  $J_\gamma = 0$ .*

*Proof.* Note that  $1_C(t, t') = 1_{L_t}(t')$ , where

$$L_t = \bigcup_1^k L_{t_i}^i.$$

Therefore

$$J_\gamma = \int \gamma(L_t) \gamma(dt) = \sum_1^k \int \gamma(L_{t_i}^i) \gamma(dt) = \sum_1^k \int \gamma(L_s^i) \gamma_i(ds), \quad (3.4)$$

where  $\gamma_i$  is the projection of  $\gamma$  on the  $i$ th coordinate axis. The integrand vanishes outside the set  $A_i$  defined by (3.1). Therefore

$$\int \gamma(L_s^i) \gamma_i(ds) = \sum_{s \in A_i} \gamma(L_s^i)^2 \quad (3.5)$$

and

$$J_\gamma = \sum_L \gamma(L)^2, \quad (3.6)$$

where  $L$  runs over  $\mathbb{L}(\gamma)$ . ■

**3.3. Proof of Theorem 1.4.** If  $\nu(B^c \times E^k) = 0$ , then  $A_\nu(B) = 0$   $P_\mu$ -a.s. by (1.22) which implies (a).

To prove (b), we consider the projection  $\gamma$  of  $\nu$  on  $\mathbb{R}_+^k$  and function  $\varphi(t, t') = 1_C(t, t')$  where  $C$  is given by (3.3). Note that

$$J_\gamma = (\nu \times \nu)_\varphi(\mathbb{R}^{2k}), \quad (3.7)$$

$$J_{A_\nu} = \int_{T^2} \varphi(t, t') A_\nu(dt) A_\nu(dt'). \quad (3.8)$$

If  $\nu$  is diffuse, then  $\gamma$  is diffuse and  $J_\gamma = 0$  by Lemma 3.2. By (3.7), (1.23) and (3.8),  $P_\mu$ -a.s.,  $J_{A_\nu} = 0$  and  $A_\nu$  is diffuse by Lemma 3.2.

Let us prove (c). We have  $\nu = \nu_1 + \nu_2$  where  $\nu_1$  is the restriction of  $\nu$  to  $\mathcal{L}(\nu) \times E^k$  and  $\nu_2[\mathcal{L}(\nu) \times E^k] = 0$ . If Condition (1.11) holds for  $(\mu, \nu)$ , then it holds for  $(\mu, \nu_1)$  and for  $(\mu, \nu_2)$ . By Theorem 1.1, there exist PLA functionals  $A_{\nu_1}$  and  $A_{\nu_2}$  with determining set  $\mathcal{M}^*$  such that  $A_\nu = A_{\nu_1} + A_{\nu_2}$ . Let  $\mu \in \mathcal{M}^*$ . It follows from (a) that,  $P_\mu$ -a.s.,  $A_{\nu_1}$  is concentrated on  $\mathcal{L}(\nu) \times E^k$  and therefore  $\mathbb{L}_{A_{\nu_1}} \subset \mathbb{L}_\nu$  by Lemma 3.1. Measure  $\nu_2$  is diffuse and  $A_{\nu_2}$  is diffuse  $P_\mu$ -a.s. by (b). Hence  $\mathbb{L}_{A_\nu} = \mathbb{L}_{A_{\nu_1}} \cup \mathbb{L}_{A_{\nu_2}} = \mathbb{L}_{A_{\nu_1}} \subset \mathbb{L}_\nu$   $P_\mu$ -a.s. ■

#### 4. BIBLIOGRAPHICAL NOTES

**4.1.** Additive functionals of higher order related to the self-intersections of the Brownian motion appeared first in connection with the Symanzik's model of the Euclidean quantum field (see, e.g., the Survey of Literature in [5].) A general theory of functionals of order  $k$  is developed in [2]. We start from functionals of the form

$$A(B) = \int_B \rho(\xi_{t_1}^1, \dots, \xi_{t_k}^k) dt_1 \cdots dt_k, \quad (4.1)$$

where  $(\xi^1, P^1), \dots, (\xi^k, P^k)$  are independent time-reversible Markov processes. To simplify presentation, we assume here that  $\xi^1, \dots, \xi^k$  are continuous processes in a Polish space  $E$  [a more general class of processes was

considered in [2]]. Measures  $m_i(dx) = P^i\{\xi_t \in dx_i\}$  do not depend on  $t$ . Let  $p_t^i(x, y)$  be a transition density for  $\xi_t^i$  relative to  $m_i$ . Put

$$g^i(x, y) = \int_0^\infty e^{-t} p_t^i(x, y) dt \quad \text{for } x, y \in E,$$

and set

$$\langle \eta, \eta \rangle = \int_{E^k \times E^k} \eta(dx) \eta(dy) \prod_1^k g_1^i(x_i, y_i)$$

for every measure  $\eta$  on  $E^k$ .

It was proved in [2] that to every finite measure  $\eta$ , subject to the condition  $\langle \eta, \eta \rangle < \infty$ , there corresponds an additive functional  $A_\eta$  such that

$$P[A_\eta(T_+)^2] = \langle \eta, \eta \rangle,$$

where  $P = P^1 \times \dots \times P^k$  and  $T_+ = \{t: 0 < t_1 < \dots < t_k\}$ . Functional (4.1) corresponds to measure  $\eta(dx_1, \dots, dx_k) = \rho(x_1, \dots, x_k) m_1(dx_1) \dots m_k(dx_k)$ . A general functional  $A_v$  can be obtained as an  $L^2(P)$ -limit of functionals of the form (4.1). All measures  $A_\eta$  are,  $P$ -a.s., diffuse in the sense of Section 1.4.

**4.2.** Investigation of additive functionals of several independent Markov processes was continued by Fitzsimmons and Salisbury [9] who established an important connection between an energy defined in terms of functionals  $A_\eta$  and hitting probabilities for  $(\xi_{t_1}, \dots, \xi_{t_k})$ .

**4.3.** Additive functionals of order  $k$  of a single Markov process  $\xi = (\xi_t, P)$  were studied (under the name "multiple path integrals") in [3]. We assume that, for all  $0 < t_1 < \dots < t_k$ ,

$$P\{\xi_{t_1} \in dx_1, \dots, \xi_{t_n} \in dx_n\} = p(t_1, x_1; \dots; t_k, x_k) m_{t_1}(dx_1) \dots m_{t_k}(dx_k),$$

where

$$m_t(dx) = P\{\xi_t \in dx\}.$$

We say that  $\nu$  is the spectral measure of a functional  $A$  if

$$\begin{aligned} P \int f(t_1, \xi_{t_1}; \dots; t_k, \xi_{t_k}) A(dt) \\ = \int f(t_1, x_1; \dots; t_k, x_k) p(t_1, x_1; \dots; t_k, x_k) \nu(dt_1, dx_1; \dots; dt_k, dx_k) \end{aligned}$$

for every positive Borel function  $f$ . Let  $p(s, x; t, y) = p_{t-s}(x, y)$  satisfy (1.4) and let  $g_u(x, y)$  be defined by Formula (1.35). For every measure  $\eta$ , subject to the condition

$$\int_{E^k \times E^k} \eta(dx) \eta(dy) \prod_1^k g_u(x_i, y_i) < \infty,$$

we construct in [3] a functional  $A_\eta$  with the spectral measure

$$\eta(dx_1, \dots, dx_k) dt_1, \dots, dt_k.$$

[Functional

$$A(B) = \int_B \rho(t_1, \xi_{t_1}; \dots; t_k, \xi_{t_k}) dt_1 \cdots dt_k$$

corresponds to the measure

$$\rho(t_1, x_1; \dots; t_k, x_k) m_{t_1}(dx_1) \cdots m_{t_k}(dx_k).]$$

**4.4.** Local times  $L_c$  for the super-Brownian motion were introduced by Iscoe in [11]. Self-intersection local times  $L^k$  are closely related to  $k$ -multiple points of superprocesses studied by Perkins [12]. The corresponding PLA functionals were constructed in [4].

**4.5.** The continuity properties for PLA functionals of superprocesses stated in Theorem 1.4 were known before only in the case  $k=1$  (see Theorem 7.2 in [6]).

## REFERENCES

1. E. B. Dynkin, Minimal excessive measures and functions, *Trans. Amer. Math. Soc.* **258** (1980), 217–244. [Reprinted in: E. B. Dynkin, “Markov Processes and Related Problems of Analysis,” London Math. Soc. Lecture Notes, Ser. 54, Cambridge Univ. Press, Cambridge, UK, 1982]
2. E. B. Dynkin, Additive functionals of several time-reversible Markov processes, *J. Funct. Anal.* **42** (1981), 64–101.
3. E. B. Dynkin, Multiple path integrals, *Adv. Appl. Math.* **7** (1986), 205–219.
4. E. B. Dynkin, Representation for functionals of superprocesses by multiple stochastic integrals, with applications to self-intersectional local times, *Astérisque* **157–158** (1988), 147–171.
5. E. B. Dynkin, Self-intersection gauge for random walks and for Brownian motion, *Ann. Probab.* **16** (1988), 1–57.
6. E. B. Dynkin, Superprocesses and their linear additive functions, *Trans. Amer. Math. Soc.* **314** (1989), 255–282.

7. E. B. Dynkin and R. K. Gettoor, Additive functionals and entrance laws, *J. Funct. Anal.* **62** (1985), 221–265.
8. E. B. Dynkin and S. E. Kuznetsov, Determining functions of Markov processes and corresponding dual regular classes, *Dokl. Akad. Nauk USSR* **414** (1974), 25–28. [English translation in *Soviet Math.—Doklady* **15** (1974)]
9. P. J. Fitzsimmons and T. S. Salisbury, Capacity and energy for multiparameter Markov processes, *Ann. Inst. Henri Poincaré, Probabilités et Statistiques* **25** (1989), 325–350.
10. A. Friedman, “Partial Differential Equations of Parabolic Type,” Prentice–Hall International, Englewood Cliffs, NJ, 1964.
11. I. Iscoe, Ergodic theory and local occupation time for measure-valued branching Brownian motion, *Stochastics* **18** (1986), 197–243.
12. E. Perkins, Polar sets and multiple points for super-Brownian motion, *Ann. Probab.* **18** (1990), 453–491.